

From the sine–Gordon field theory to the Kardar–Parisi–Zhang growth equation

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We unveil a remarkable connection between the sine–Gordon quantum field theory and the Kardar–Parisi–Zhang (KPZ) growth equation. We find that the non-relativistic limit of the two point correlation function of the sine–Gordon theory is related to the generating function of the height distribution of the KPZ field with droplet initial conditions, i.e. the directed polymer free energy with two endpoints fixed. As shown recently, the latter can be expressed as a Fredholm determinant which in the large time separation limit converges to the GUE Tracy–Widom cumulative distribution. Possible applications and extensions are discussed.

The sine–Gordon (sG) model and its cousin, the sinh–Gordon (shG) model, are paradigmatic integrable quantum field theories with countless applications in condensed matter physics (see e.g. [1] as a review). A lot is known in particular about their spectra in terms of many-particle scattering states and about matrix elements of local operators, the so-called form factors [2, 3]. This allowed very striking predictions for experimentally relevant systems (such as spin chains and ladders [1]) which in the scaling limit are described by these massive field theories. While the shG model contains only a single type of particle of mass M , the excitation spectrum of the sG model exhibits solitons as well as “breathers” B_m that can be viewed as bound states of m ‘particles’ or, alternatively, as soliton-antisoliton bound states. The two models are also related by an analytic continuation of the coupling constant.

As a recent experimentally relevant application, the non-relativistic limit (NRL) of the shG model was considered in Ref. [4], where it was shown that in a double scaling limit (i.e. taking a NRL while taking the shG coupling to zero) the Lieb–Liniger (LL) model [5] is recovered with *repulsive* interactions. This procedure allowed the analytic calculation of some previously unknown local expectation values [4, 6]. Taking the same double scaling limit of the sG model instead gives the Lieb–Liniger (LL) model [5] with *attractive* interactions, and indeed it is known that the scattering phases of the two models do coincide [7]. However, this procedure has not yet been explored to obtain expectation values of measurable observables.

In an a priori completely different context, there has been much progress in finding exact solutions to the 1D Kardar–Parisi–Zhang (KPZ) equation [8–22]. Some of these approaches used the mapping onto the directed polymer (DP) model, and from there, using the replica method, onto the LL model of bosons with attractive interactions [23]. These bosons form bound states called strings in the large system size limit [24], and summation over these string states has allowed for the calculation of the probability distribution function (PDF) of the KPZ height field for various initial conditions [9, 10, 13, 15, 16].

Some of these predictions have been tested experimentally [25].

The aim of this paper is to unveil a connection between the KPZ or DP models and the NRL of the sG quantum field theory which arises because, as we mentioned above, in the NRL the sG model reproduces the *attractive* LL model. While m , characterizing the breathers, is a bounded integer for sG, it becomes unbounded in the NRL and reproduces the bound states of the LL model, the so-called string states. Working out the details, we show that the connection is remarkably simple and, in particular, we find that the two point correlation function of exponential fields (vertex operators) in the sG model encodes the information of the PDF of the height field in KPZ.

KPZ equation and Lieb–Liniger model: Let us recall some facts about the KPZ equation in one space dimension. It describes the stochastic growth of an interface of height $h(x, t)$, as a function of time as

$$\partial_t h = \nabla^2 h + (\nabla h)^2 + \eta \quad (1)$$

in dimensionless units, in the presence of white noise $\eta(x, t)\eta(x', t') = 2\bar{c}\delta(x - x')\delta(t - t')$. Here we focus on the so-called droplet, or sharp-wedge, initial condition, where $h(x, t = 0) = -w|x|$ with $w \rightarrow +\infty$. The height field $h(x, t)$ can be written as $h = \ln Z$ where Z is the partition sum of a fixed endpoint DP, i.e. paths in the x, t plane with endpoints fixed at $(0, 0)$ and (x, t) , directed along the time direction. In the KPZ context one introduces the generating function

$$g(u) = \sum_{n=0}^{\infty} \frac{(-u)^n}{n!} \overline{Z^n}, \quad (2)$$

which is a series representation for the Laplace transform $\frac{1}{e^{-uZ}}$ of the PDF of Z . In the large time limit the (one-point) fluctuations of $h(x, t)$ are expected to grow as $t^{1/3}$.

To study this limit it is convenient to write $h(x = 0, t) \simeq vt + 2^{2/3}\lambda\chi(t)$ where $\lambda = (\bar{c}^2 t/4)^{1/3}$ and $\chi(t)$ is an $O(1)$ random variable. We also define $u = e^{-\lambda s}$, since then $\tilde{g}(s) = g(u = e^{-\lambda s}) \rightarrow \text{Prob}(\chi(t) < s)$ for $\lambda \rightarrow \infty$,

hence the generating function $\tilde{g}(s)$ is directly the cumulative distribution function (CDF) of the height. The important finding of Refs. [9, 10] is that $\tilde{g}(s)$ can be expressed as a Fredholm determinant at all times (see below) and that it converges for large time to $F_2(s)$, the GUE Tracy Widom CDF [26]. These results were obtained exploiting the property that the moments \overline{Z}^n can be written as the diagonal propagator of an n -particle Bose gas with the Hamiltonian

$$H_{LL} = \frac{1}{2M} \sum_{\alpha=1}^n \partial_{x_\alpha}^2 - 2\bar{c} \sum_{\alpha < \beta} \delta(x_\alpha - x_\beta), \quad (3)$$

i.e. the attractive LL model which is integrable via the Bethe ansatz [27]. More precisely, \overline{Z}^n reads

$$\overline{Z}^n = \langle \vec{x}_0 | e^{-tH_{LL}} | \vec{x}_0 \rangle = \sum_{\mu_n} \frac{|\langle \vec{x}_0 | \mu_n \rangle|^2}{||\mu_n||^2} e^{-tE_{\mu_n}}, \quad (4)$$

where μ_n are the n -boson eigenstates of H_{LL} with eigenenergies E_{μ_n} , $\vec{x}_0 = (x_0 \dots x_0)$, so $\langle \vec{x}_0 | \mu_n \rangle$ is the real space wave function evaluated at coinciding particle positions in x_0 . Evaluating this sum over Bethe states in the infinite system size limit is not trivial and was performed in Refs. [9, 10]. Here we show that the same sum can be retrieved rather simply, directly in the infinite system limit, from known results in the sG field theory, without the need to manipulate Bethe states. Besides its interest for the LL physics, it also raises hope that other interesting quantities in KPZ could be calculated using the sG model.

Sine-Gordon model: The sG model is a relativistically invariant integrable quantum field theory in $(1+1)$ dimension defined by the *Euclidean* (imaginary time) Lagrangian density

$$\mathcal{L}_{sG} = \frac{1}{2c_l^2} (\partial_t \phi)^2 + \frac{1}{2} (\partial_x \phi)^2 - \frac{m_0^2 c_l^2}{\beta^2} (\cos(\beta \phi) - 1), \quad (5)$$

where $\phi(x, t)$ is a real scalar field, c_l is the speed of light, m_0 the bare mass, β the coupling constant, and we set $\hbar = 1$. The renormalized (physical) coupling constant and mass are

$$\hat{\alpha} = \frac{c_l \beta^2}{8\pi - c_l \beta^2}, \quad (6)$$

$$M^2 = m_0^2 \frac{\sin(\pi \hat{\alpha})}{\pi \hat{\alpha}}. \quad (7)$$

The spectrum of the theory contains several kinds of particles. The ‘fundamental’ one is called 1-breather which has relativistic dispersion relation with energy and momentum

$$E(\theta) = M c_l^2 \cosh(\theta), \quad p(\theta) = M c_l \sinh(\theta), \quad (8)$$

where θ is the rapidity. The spectrum also contains solitons as well as m -breathers which are bound states of

particles. We will need below the dispersion relation of the m -breather. Its total energy and momentum as a function of its center of mass rapidity θ can be obtained by introducing $\theta^a = \theta - \frac{2a-1-m}{2} i\pi \hat{\alpha}$ and writing

$$E_m(\theta) = M c_l^2 \sum_{a=1}^m \cosh(\theta^a) = M_m c_l^2 \cosh(\theta), \quad (9)$$

$$P_m(\theta) = M c_l \sum_{a=1}^m \sinh(\theta^a) = M_m c_l \sinh(\theta), \quad (10)$$

leading to the same dispersion relation as for 1-breathers but with a different mass M_m [28]

$$M_m = M \frac{\sin(m\pi \hat{\alpha}/2)}{\sin(\pi \hat{\alpha}/2)}. \quad (11)$$

Note that the particle content m of the breather is bounded in the relativistic sG model, namely $m_{\max} = [1/\hat{\alpha}]$. Due to its integrability, the sG model supports diffractionless factorized scattering and the exact 2-particle S-matrix is known [3, 29]. Based on this, matrix elements of exponential vertex operators between scattering states, i.e. form factors have been computed exactly [2, 30, 31].

Double non-relativistic limit and LL model: Here we are interested in the (double) NRL defined as [4]

$$c_l \rightarrow +\infty, \quad \beta \rightarrow 0, \quad \beta c_l = 4\sqrt{\bar{c}}, \quad (12)$$

with \bar{c} fixed and finite. Hence the renormalized coupling tends to zero as

$$\hat{\alpha} \simeq \frac{c_l \beta^2}{8\pi} = \frac{2\bar{c}}{\pi c_l} \rightarrow 0. \quad (13)$$

In this limit the dispersion relation of the particles becomes non-relativistic as

$$p \simeq M c_l \theta \equiv \lambda, \quad E \simeq M c_l^2 + \frac{\lambda^2}{2M}, \quad (14)$$

where λ is the usual (non-relativistic) rapidity, i.e. the quasi-momentum. Importantly, this double limit establishes a connection between the exact S-matrices and the form factors of the two models. At the level of the fields the correspondence can be written in the double limit as

$$\phi(x, t) = \frac{1}{\sqrt{2m_0}} \left[\Psi(x, t) e^{-m_0 c_l^2 t} + \Psi^\dagger(x, t) e^{m_0 c_l^2 t} \right], \quad (15)$$

where ϕ is the sG field and Ψ^\dagger creates a non-relativistic particle (i.e. the LL boson). Plugging (15) into (5) leads, upon expansion and neglecting highly oscillating terms, to the non-linear Schrödinger Hamiltonian

$$H_{LL} = \int dx \left(\frac{1}{2M} \nabla \Psi^\dagger \nabla \Psi - \bar{c} \Psi^\dagger \Psi^\dagger \Psi \Psi \right), \quad (16)$$

which is the second quantized form of the attractive LL Hamiltonian (3). This limit procedure was first shown

for the shG model in Ref. [4], leading to the repulsive LL model with interaction parameter $c = -\bar{c} > 0$. Here the same method shows that $\bar{c} > 0$ emerges as the coupling constant of the attractive LL model. In fact, at the level of the single particle states a lot can be deduced by analytical continuation from shG to sG ($\beta \rightarrow i\beta$). This technique was used to study a highly excited gas-like state of the attractive LL model, the super Tonks-Girardeau gas [32]. However, the many-particle states in general are quite different in the repulsive ($\bar{c} < 0$) and attractive ($\bar{c} > 0$) cases. Consider the energy of the sG m -breather (9-11) in the double NRL:

$$E_m(\theta) \simeq Mmc_l^2 + \frac{\bar{c}^2}{24M}(m - m^3) + m\frac{p^2}{2M}, \quad (17)$$

while the momentum is $P_m(\theta) = mp$, where we have scaled $\theta = p/(Mc_l)$ and neglected terms $O(1/c_l)$. Apart from the rest energy, these are exactly the total energies and momenta of the m -string states of the LL model in the limit of infinite system size [24]. The correspondence goes further and indeed also the scattering phases coincide, as pointed out in [7]. Finally, in the NRL the mass of the solitons/antisolitons $M_s = 8m_0^2/\beta^2$ [3] diverges, hence they disappear from the spectrum and we can neglect them. We are then left with an infinite number of breather modes as $\hat{\alpha} \rightarrow 0$ corresponding to the LL strings. The form factors of the LL model can also be

obtained through the NRL from the breather form factors of the sG model, paralleling the calculation for the shG model [4, 33]. In the remainder of the paper we will set $M = 1/2$ by a choice of units, as customary in the LL model.

From sG correlations to the KPZ/DP model: Let us consider the two-point correlator of the exponential field in the Euclidean sG model (i.e. in imaginary time)

$$G(\tilde{k}, t) = \langle 0 | e^{i\tilde{k}\phi(0,t)} e^{-i\tilde{k}\phi(0,0)} | 0 \rangle, \quad (18)$$

as well as the reduced correlation [38]

$$\tilde{G}(\tilde{k}, t) = G(\tilde{k}, t) / |\langle e^{i\tilde{k}\phi} \rangle|^2, \quad (19)$$

which is defined to equal unity at $t \rightarrow \infty$. Here and below we denote the vacuum expectation value of the exponential field, as $\langle e^{i\tilde{k}\phi} \rangle = \langle 0 | e^{i\tilde{k}\phi(0,0)} | 0 \rangle$.

The Lehmann formula [1–3] expresses quite generally such a ground state expectation value (here in the vacuum) in terms of the form factors of the excitations of the theory. At this stage we do not yet consider the NRL but, for simplicity, we ignore the solitons states (which will be justified only in that limit). It thus takes the form of a sum over states with arbitrary number n_s of breathers

$$G(\tilde{k}, t) \simeq \sum_{n_s=0}^{\infty} \frac{1}{n_s!} \prod_{j=1}^{n_s} \sum_{m_j=1}^{m_{max}} \int \frac{d\theta_1}{2\pi} \dots \frac{d\theta_{n_s}}{2\pi} |\langle 0 | e^{i\tilde{k}\phi(0,0)} | B_{m_1}(\theta_1) \dots B_{m_{n_s}}(\theta_{n_s}) \rangle|^2 e^{-\sum_{j=1}^{n_s} E_{m_j}(\theta_j)|t|}, \quad (20)$$

where each breather of type m_j has rapidity θ_j and particle content m_j .

The form factors of the breathers can be obtained from those of the particles $B_1(\theta)$. Hence let us start with the particle states, i.e. we temporarily restrict to $m_j = 1$ in (20). Their form factors are known in sG [34] and can be obtained from the shG ones [35] via analytical continuation to imaginary coupling as

$$F_n^k(\theta) = \langle 0 | e^{ik\beta\phi} | \theta_1 \dots \theta_n \rangle = \langle e^{ik\beta\phi} \rangle \frac{\sin(k\pi\hat{\alpha})}{\sin(\pi\hat{\alpha})} (2i)^n \times \left(\frac{\sin(\pi\hat{\alpha})}{F_{min}(i\pi)} \right)^{n/2} \det M_n(k) \prod_{j<l} \frac{F_{min}(\theta_j - \theta_l)}{e^{\theta_j} + e^{\theta_l}}, \quad (21)$$

where $|\theta_1 \dots \theta_n\rangle = |B_1(\theta_1) \dots B_1(\theta_n)\rangle$. Here we introduced the matrix (with $j, l = 1, \dots, n-1$)

$$[M_n(k)]_{j,l} = \frac{\sin[(j-l+k)\pi\hat{\alpha}]}{\sin(\pi\hat{\alpha})} \sigma_{2j-l}^{(n)}, \quad (22)$$

for $n \geq 2$, where $\sigma_j^{(n)}$ is the j -th elementary symmetric polynomial of the variables $\{e^{\theta_1}, \dots, e^{\theta_n}\}$, and $\det M_1(k) = 1$. The minimal form factor is given by

$$F_{min}(\theta) = \mathcal{N} e^{-4 \int_0^\infty \frac{dt}{t} \frac{\sinh(\frac{t}{2}\hat{\alpha}) \sinh(\frac{t}{2}(1+\hat{\alpha}))}{\sinh(t) \cosh(\frac{t}{2})} \sin^2\left(\frac{t(i\pi-\theta)}{2\pi}\right)}, \quad (23)$$

where $\mathcal{N} = F_{min}(i\pi) = e^{\frac{1}{\pi} \int_0^{\pi\hat{\alpha}} dt \frac{t}{\sin(t)}} / \cos\left(\frac{\pi\hat{\alpha}}{2}\right)$. The minimal form factor satisfies the exact relation [35]

$$F_{min}(i\pi + \theta) F_{min}(\theta) = \frac{\sinh \theta}{-\sinh(i\hat{\alpha}\pi) + \sinh \theta}. \quad (24)$$

We now carefully take the (double) non-relativistic limit of the form factors. To obtain a non-trivial limit we need to let $k \rightarrow \infty$ while $\beta \rightarrow 0$ with $\tilde{k} = k\beta$ fixed and finite. We also scale the rapidities as $\theta_j = 2p_j/c_l$ (we recall $M = 1/2$). Given that $\mathcal{N} \rightarrow 1$ as $\hat{\alpha} \rightarrow 0$, using Eq. (24) we have in the NRL

$$F_{min}(\theta) \rightarrow \frac{\theta}{-i\hat{\alpha}\pi + \theta}. \quad (25)$$

The most complicated term in the form factor (21) is $\det M_n$ which immensely simplifies in the NRL because one can neglect the $i - j \ll k$ in the matrix M_n to get

$$\det [M_n(\tilde{k}/\beta)] \simeq \left(\frac{\sin(k\pi\hat{\alpha})}{\sin(\pi\hat{\alpha})} \right)^{n-1} \det \sigma_{2j-l}^{(n)}, \quad (26)$$

and $\det \sigma_{2j-l}^{(n)} \rightarrow 2^{n(n-1)/2}$ in the NRL. Finally, in the NRL we also have the simple relation

$$\frac{\sin(k\pi\hat{\alpha})}{\sin(\pi\hat{\alpha})} \rightarrow \frac{\sin(\frac{\sqrt{c}}{2}\tilde{k})}{\pi\hat{\alpha}}. \quad (27)$$

All these relations lead for the most complicated parts of the form factor to the remarkably simple limit

$$\begin{aligned} \frac{\sin(k\pi\hat{\alpha})}{\sin(\pi\hat{\alpha})} 2^n \left(\frac{\sin(\pi\hat{\alpha})}{F_{min}(i\pi)} \right)^{n/2} \det M_n(k) \prod_{j<l} \frac{1}{e^{\theta_j} + e^{\theta_l}} \\ \rightarrow \left(\frac{c_l}{2} \right)^{n/2} \left[\frac{2}{\sqrt{c}} \sin\left(\frac{\sqrt{c}}{2}\tilde{k}\right) \right]^n. \end{aligned} \quad (28)$$

Plugging this in (20) and taking into account the Jacobian of the variable change from θ 's to p 's, we find for the $m_j = 1$ contribution

$$\begin{aligned} G(\tilde{k}, t) \simeq |\langle e^{i\tilde{k}\phi} \rangle|^2 \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{2}{\sqrt{c}} \sin\left(\frac{\sqrt{c}}{2}\tilde{k}\right) \right]^{2n} e^{-nMc_l^2|t|} \\ \int \frac{dp_1}{2\pi} \cdots \frac{dp_n}{2\pi} \prod_{j<l} \frac{(p_j - p_l)^2}{c^2 + (p_j - p_l)^2} e^{-\sum_j p_j^2|t|}, \end{aligned} \quad (29)$$

which is a sum of positive contributions, as it should, since it comes from the Lehmann formula.

Let us now generalize the derivation to arbitrary m -breather states. Fusion relations relate the form factors of breathers to the ones of n particles F_n as follows. Let us recall that the rapidities can be written as $\theta_j^a = \theta_j - \frac{2a-1-m_j}{2}i\pi\hat{\alpha}$. Then, recalling $n = \sum_j m_j$, we have [34]

$$\begin{aligned} \langle 0 | e^{ik\beta\phi} | B_{m_1}(\theta_1) \cdots B_{m_{n_s}}(\theta_{n_s}) \rangle = \\ \prod_{j=1}^{n_s} \gamma_{m_j} F_n(\{\theta_1^{a_1}\}_{a_1=1,\dots,m_1}, \dots, \{\theta_{n_s}^{a_{n_s}}\}_{a_{n_s}=1,\dots,m_{n_s}}), \end{aligned} \quad (30)$$

where the NRL of γ_m given in [34] is

$$\gamma_m = (\pi\hat{\alpha})^{(m-1)/2} \Gamma[m] \sqrt{m}. \quad (31)$$

Note that $\theta_j^a \rightarrow \frac{\pi\hat{\alpha}}{c}[p_j - \frac{i\tilde{c}}{2}(2a-1-m_j)]$ which coincide with the string rapidities [24].

In taking the NRL of the form factor F_n , only the term with F_{min} changes compared to the previous case, leading to

$$\prod_{1 \leq j < l < n} |F_{min}(\theta_j - \theta_l)|^2 = \Phi[p, m] \prod_{j=1}^{n_s} |F[m_j]|^2, \quad (32)$$

where

$$\Phi[p, m] = \prod_{1 \leq j < l \leq n_s} \frac{4(p_i - p_j)^2 + \tilde{c}^2(m_i - m_j)^2}{4(p_i - p_j)^2 + \tilde{c}^2(m_i + m_j)^2}, \quad (33)$$

$$F[m] = \prod_{1 \leq a < b \leq m} \frac{b-a}{b-a+1} = \frac{1}{m!}. \quad (34)$$

Putting everything together we finally obtain

$$\begin{aligned} G(\tilde{k}, t) \simeq |\langle e^{i\tilde{k}\phi} \rangle|^2 \sum_{n_s=0}^{\infty} \frac{\tilde{c}^{n-n_s}}{n_s!} \prod_{j=1}^{n_s} \sum_{m_j=1}^{+\infty} \left[\frac{2}{\sqrt{c}} \sin\left(\frac{\sqrt{c}}{2}\tilde{k}\right) \right]^{2m_j} \\ \prod_{j=1}^{n_s} \int \frac{dp_j}{2\pi m_j} e^{-m_j Mc_l^2 t - \frac{\tilde{c}^2}{12}(m_j^3 - m_j)t - m_j p_j^2 t} \Phi[p, m]. \end{aligned} \quad (35)$$

Now we can compare this with the expression for the moments of the partition sum in the KPZ/DP problem. The calculation of the averaged moments (4) was performed in Ref. [9] and found to take exactly the same expression as above (compare with Eq. (9) of Ref. [9]). Hence we find that the reduced correlation can be written as

$$\tilde{G}(\tilde{k}, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{2}{\sqrt{c}} \sin\left(\frac{\sqrt{c}}{2}\tilde{k}\right) \right]^{2n} \overline{Z}^n e^{-nMc_l^2 t}, \quad (36)$$

showing that there is a relation between the two-point correlation in sG and the moments in the KPZ/DP problem. Furthermore, the sG correlation takes the same form as the KPZ/DP generating function, hence one can also write

$$\tilde{G}(\tilde{k}, t) = g(u), \quad (37)$$

where $u = -[\frac{2}{\sqrt{c}} \sin(\frac{\sqrt{c}}{2}\tilde{k})]^2 e^{-Mc_l^2 t}$. Interestingly, by analytic continuation $i\tilde{k} \rightarrow \tilde{k}$ one also finds

$$\begin{aligned} \frac{\langle 0 | e^{\tilde{k}(\phi(0,t) - \phi(0,0))} | 0 \rangle}{\langle e^{\tilde{k}\phi} \rangle^2} = \\ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{2}{\sqrt{c}} \sinh\left(\frac{\sqrt{c}}{2}\tilde{k}\right) \right]^{2n} \overline{Z}^n e^{-nMc_l^2 t} = g(u), \end{aligned} \quad (38)$$

where now $u = [\frac{2}{\sqrt{c}} \sinh(\frac{\sqrt{c}}{2}\tilde{k})]^2 e^{-Mc_l^2 t}$ is a positive number, thereby making the connection closer. Since the KPZ generating function obeys $0 < g(u) < 1$, it implies that both sides of the above equation are now positive numbers in the interval $[0, 1]$, increasing with \tilde{k} . Note that although the operator in the left hand side of Eq. (38) may not be formally defined in the sG field theory, in the Euclidean version it takes the meaning of a canonical statistical mechanics average, with a discretization and regularization at small and large scale (as one would do in a numerical simulation). While the two correlations in Eq. (38) may be singular as the regularizations are

removed, their ratio should be a well defined number in the interval $[0, 1]$.

Having shown that the NRL of the sG correlation contains information about the PDF of the KPZ field, we can ask whether a tighter physical connection exists. Let us recall the expression of the generating function as a Fredholm determinant (FD) obtained in [9]:

$$\tilde{g}(s) = \text{Det}[1 + P_0 K_s P_0], \quad (39)$$

where P_0 is the projector on $[0, +\infty[$ and the kernel can be written as

$$K_s(v, v') = - \int \frac{dk}{2\pi} dy Ai(y + k^2 + s + v + v') \frac{e^{\lambda y - ik(v-v')}}{1 + e^{\lambda y}}, \quad (40)$$

where $\lambda \sim t^{1/3}$ was defined above. Let us also recall that at large time this generating function $\tilde{g}(s)$ converges to $F_2(s)$, the GUE Tracy Widom CDF [26]. So it is interesting that this FD contains the information about the precise time dependence of the coefficients of each power of $e^{-Mc_l^2 t}$ in the decay of the sG correlation function, these coefficients being proportional to the KPZ/DP moments $\overline{Z^n}$. Note however, that because of these fast decaying exponentials, there is no point-wise convergence as $c_l \rightarrow \infty$ in (36) and (38) and, at this stage, there is no *direct* correspondence between a sG observable and KPZ generating function. An outstanding question is thus how far can this correspondence be pushed and whether one can construct sG observables with an even closer relation to KPZ.

One last interesting point relates to the so-called moment problem in the continuum KPZ/DP problem, i.e. to the fact that the growth of the integer moments $\overline{Z^m}$ as a function of m at fixed t is too rapid (i.e. $\ln \overline{Z^m} \sim m^3 t$) to guarantee a unique solution for the PDF for $h = \ln Z$. While this is still an open question in the mathematics community, it is usually circumvented by resorting to discrete models (such as TASEP, see e.g. [11]) which do reproduce continuum KPZ in some limit and do not suffer from the same problem. We point out here that the relativistic sG theory can provide yet another interesting regularization of the moment problem because the number of breather types $[1/\hat{\alpha}]$ is bounded until the NRL is taken.

Overlaps and propagators in the LL model: One can ask more precisely how can the sG theory retrieve more detailed information about the attractive LL model hereby helping solve KPZ problems. For example a related fundamental quantity (both for the LL model and the KPZ growth) is the overlap

$$G(|\psi_1\rangle, |\psi_0\rangle; t) = \langle \psi_1 | e^{-tH_{LL}} | \psi_0 \rangle \quad (41)$$

between more general initial $|\psi_0\rangle$ and final states $|\psi_1\rangle$. Examples are (i) KPZ with flat initial condition, which requires the overlap with an initial uniform state and

whose solution was obtained in [13] (ii) the imaginary-time propagator for arbitrary positions [36, 37]. Here we have shown that in the NRL the diagonal propagator can be retrieved as [39]

$$\tilde{G}(\tilde{k}, t)|_{e^{-nMc_l^2 t}} = \frac{1}{n!} \left[\frac{2}{\sqrt{c}} \sin\left(\frac{\sqrt{c}}{2} \tilde{k}\right) \right]^{2n} G(|\vec{x}_0\rangle, |\vec{x}_0\rangle; t). \quad (42)$$

It would be interesting to obtain more general overlaps via this sG correspondence which we leave for future investigations.

Relation (42) can be understood in the following way. The diagonal propagator, $\langle \Psi^n(t) \Psi^{\dagger n}(0) \rangle$ can be extracted from the NRL of the sG correlator $\langle \phi^n(t) \phi^n(0) \rangle$ where the operator ϕ^n can be obtained naively from the n th order of the series expansion of $e^{ik\phi}$ in powers of k . However, it turns out that the ϕ^n operator obtained in this way has non-zero matrix elements between states having any number of particles. These form factors survive in the NRL meaning that in the limit one cannot recover the Ψ^n operator. It turns out that certain linear combinations of different powers of ϕ lead to Ψ^n in the NRL. Quite interestingly, these combinations in terms of k give just the sine factor in Eq. (42), so this formula, quite miraculously, automatically takes care of this operator mixing.

Conclusion. In this paper we have shown yet another method to calculate the PDF for the KPZ growth equation with the narrow wedge initial condition exploiting the NRL limit of sG field theory. The obtained result fully agrees with previous derivations [9–12]. On the one hand this result is a useful check of our new method and of present and previous results, on the other hand this new correspondence raises the hope that it can be used to obtain yet unknown observables for the KPZ growth equation, as well as provide a new interesting regularization of the problem.

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- [38] In our normalization indeed $|\langle e^{i\vec{k}\phi} \rangle|^2 = 1$ and taking the ratio (18) is superfluous. However, this ratio is independent of normalizations and conventions and the results for this quantity are fully general.
- [39] The case $G(|\vec{x}\rangle, |\vec{y}\rangle; t)$ with $\vec{x} = (x_0 \dots x_0)$ and $\vec{y} = (y_0 \dots y_0)$ is a trivial generalization of the present formula, introducing the total momentum in Eq. (42), see e.g. [13].